CHAPTER I

INTRODUCTION

1.1 Background

In the ever-dynamic landscape of global financial markets, investors leverage an array of financial derivative assets to satisfy their investment goals. Amongst these assets, options hold a prominent position. They afford investors the prerogative to buy or sell the underlying asset at a predetermined price. The inherent nature of options empowers investors to broaden their strategy; they facilitate a range of hedging techniques and encourage speculative investments concerning the volatility of the underlying asset. This strategic latitude provided by options has been the subject of intense scrutiny since the latter part of the twentieth century. Research has delved into aspects such as the volatility risk associated with pure options portfolios [1], models for options investment centred on diversification and global hedging [2], and the application of the numerical Monte Carlo method for pricing options [3]. These studies exemplify the mathematical rigour involved in the realm of options investment strategy.

In recent years, research has shifted towards more sophisticated and nuanced methods. One novel approach, the learning random variable (LRV) technique, integrates stochastic gradient descent (SGD) optimization with Monte Carlo price simulation, producing highly accurate results even for high-dimensional models [4]. Concurrently, studies have been conducted into the performance of advanced methodologies such as support vector regression (SVR), genetic algorithm (GA), random forest (RF), and other decision tree approaches for options pricing [5]. Additionally, comparative studies have been undertaken to contrast the jump-diffusion Monte Carlo method and the classical Black-Scholes model for options pricing [6]. These developments underscore the evolving intricacy of methodologies employed in the field of options investment.

1.2 Problem Statement

The significance of options pricing in financial mathematics originates from the premise that numerous corporate liabilities can be expressed as options or their combinations [3]. Therefore, an ongoing stream of research is dedicated to devising high-performance, high-accuracy methodologies for pricing complex options. The ground-breaking paper by Black and Scholes provides a closed-form analytic solution for pricing European options with non-dividend paying underlying stock. While extensions to this analytic solution exist for continuously compounding dividend yields [7], more intricate dividend schemes typically necessitate the application of numerical methods for valuation [8, 3]. This engenders a challenging scenario wherein researchers must select from a plethora of numerical methods—such as machine learning-based approaches, binomial trees, or other numerical integration methodologies—or create a bespoke framework that aligns with their performance benchmarks.

Among the numerous numerical techniques, the Monte Carlo framework proposes a relatively straightforward method for pricing options. This is achieved by approximating the expected payoff value of the option after generating a set of price evolutions under predefined parameters and assumptions. However, the elementary form of Monte Carlo simulation is often marred by inaccuracies and a high degree of variance. This serves as the bedrock for studies aimed at refining the Monte Carlo framework. One prevalent methodology for enhancing the Monte Carlo method is classified as variance reduction techniques (VRT). The central principle of VRT involves modifying the options payoff function to primarily reduce the variance of the result. In numerous instances, reducing the variance in this manner also augments the accuracy of the result, thereby boosting the overall model's performance. Additional benefits of lower variance include a more precise confidence interval estimate and a higher rate of convergence.

Despite these advantages, the implementation of advanced VRT typically heightens the computational complexity of the simulation, implying extended simulation times. Consequently, a trade-off scenario emerges, compelling researchers to select a model that optimally aligns with their interests. Earlier research on Monte Carlo VRT primarily focused on a select few methods applied to a single or a limited subset of option styles. No existing research thoroughly examines and compares the performance of commonly used VRT across different option styles with varying underlying asset parameters. Moreover, although decision-making frameworks have been widely studied in cognitive science and utility function contexts [9, 10, 11], none have been specifically targeted at VRT model selection based on individual researcher preferences.

1.3 Research Purpose

The objectives of this research can be distilled into the following key points:

1. To propose a technical methodology for employing various Monte Carlo

variance reduction techniques (VRT) for the pricing of both ordinary and exotic options. This will include the development of mixed-model VRT, which are theoretically superior to traditional VRT.

2. To conduct a comprehensive, results-oriented analysis concerning the performance of each VRT across a broad array of option styles. This analysis aims to identify the highest-performing VRT, in a utility-neutral context, for each specific case.
3. To establish a decision-making framework offering a utility-based quantitative tool. This tool is designed to aid researchers in model selection in alignment with their specific utility specifications.

These objectives serve to improve the understanding and application of Monte Carlo variance reduction techniques in options pricing and to facilitate more effective, utility-aligned decision-making among researchers in this field.

1.4 Benefits of Research

In concurrence with the research purpose, the benefits of the research can be summarized by the following:

1. Advancing the technical methodology associated with Monte Carlo VRT, including the formulation of a mixed-model VRT.
2. Extensive exposition of the theoretical backgrounds of each VRT, which would justify the empirical results and provide valuable, credible, and tested scientific information for other researchers.
3. The application of a decision framework in this research would provide researchers an alternative view on quantitative decision-making.

1.5 List of Models

The ordinary and exotic options of interest in this research encompass:

1. Ordinary European options
2. American put option
3. Geometric price Asian options

4. Asset binary options
5. Deferred rebate options
6. Down and knock-in barrier put Options
7. Up and knock-in barrier call Options

The following VRT models of interest in this research are as follow:

1. Simple Monte Carlo simulation (SMCS)
2. Antithetic variates (AMCS)
3. Control variates (CMCS)
4. Non-parametric importance sampling (IMCS)
5. Mixed method antithetic-importance (MMAI)
6. Mixed method importance-antithetic (MMIA)

1.6 Model Assumptions

The options pricing is based on two different frameworks: The Black-Scholes (BSM) framework for the analytic solution, and the Monte Carlo simulation (MCS) framework for the numeric solution, while model selection is based on the decision framework founded on utilitarian views and VNM utility. Therefore, the resulting model assumption is the fusion these three frameworks, with some specific modifications [8, 3]:

1. Dividend payouts, if exist, are not directly computed. Instead, it is embedded in the adjusted closing price of the underlying stock.
2. Markets are frictionless, efficient, and has no taxation.
3. Investors are rational agents acting solely to maximize profit.
4. The discount rate is continuously compounded with a constant risk-free rate r, which implies that investors may borrow or invest in risk-free assets at an unlimited amount with continuously compounded risk-free rate as the interest.
5. Volatility of the underlying stock is Homoscedastic (constant volatility model).

6. The price of the underlying asset follows a geometric Brownian motion, and its returns, or the ratio of successive stock prices, is log-normally distributed.

7. The price of the options are based on risk-neutral pricing, which implies that investors are indifferent of the risk-factors associated with each options; the prices of the simulated options are solely determined by their respective discounted payoff function.

8. The preferences of each researchers can be represented accurately by the logistic function steepness parameter of their respective utility functions.

1.7 Review of Literature

The foundation of options pricing can be traced back to the seminal paper by Black and Scholes, which established a robust framework for determining options prices analytically [8]. The accuracy of VRT models, a key performance measure, can be assessed against these analytical solutions, underscoring their relevance to this research. The Black and Scholes model goes beyond merely providing explicit solutions for European call and put options; their work integral in providing the basis of risk-neutral pricing, which aids in pricing exotic options. The parameters d_1 and d_2 embody the risk-neutral probabilities, which underpin the notion that derivatives can be priced by discounting such probabilities. More specifically, $N(d_2)$ can be interpreted as the risk neutral probability that $S_T > K$, while $S \times N(d_1)$ can be understood as the risk-neutral present value of the expected asset price at expiration. This idea extends to the derivation of analytical solutions for other option styles, including barrier options. The paper also sets forth key assumptions for this research, most notably, that the returns on the underlying asset adhere to a geometric Brownian motion. Boyle's pioneering work provides a comprehensive understanding of how Monte Carlo simulations, by virtue of their sample-mean estimation and numerical integration capabilities, offer unbiased numerical price solutions [3]. Boyle introduced several strategies to curb the variance of the simulation, which led to the development of variance reduction techniques (VRT). The primary objective of VRT is to craft a function that modifies the options payoff function in Monte Carlo simulations in a manner that retains unbiasedness, reduces variance, and avoids the need for additional simulations. This idea lays the groundwork for the development of mixed-method VRT, which is the focal point of this paper.

Kroese's research presents a more contemporary and practical approach, highlighting the growing importance of Monte Carlo simulations in today's complex financial landscape, where options and other derivatives have increased in complexity [12]. Crucially, this research aligns with the primary objective of this paper: to thoroughly analyze the performance of various VRT in the context of options pricing. Finally, the ground-breaking research by Von Neumann and Morgenstern established a comprehensive mathematical foundation for utility theory [9]. They proposed a utility function that is predicated on the premise that the preferred decision maximizes expected utility value. While it is based on the concept of risky decisions, outcomes with a certain outcome can always be interpreted as degenerate lotteries. This theoretical foundation is vital for developing a rigorous, comprehensive, and accurate decision-making framework for model selection in the context of this study.

1.8 Review of Chapters

This section offers a brief summary of each chapter in this research, serving as a helpful guide for readers.

- **Chapter 1**: This introductory chapter outlines the context and justifications for the research, offering a comprehensive backdrop for the study. It outlines key assumptions, a list of models and option styles, and provides a review of relevant literature.
- **Chapter 2**: This chapter delves into the rigorous theoretical exposition that underpins this research. It covers fundamental concepts of options, offers detailed proofs and explanations of Black-Scholes solutions for various option styles, introduces Monte Carlo simulation and variance reduction techniques, explains the mathematical constructions of each VRT, and outlines the mathematical reasoning behind the decision framework used for VRT selection based on the specific preference parameters of researchers.
- **Chapter 3**: Detailing the specific methodologies employed in this research, this chapter covers additional techniques to handle the intricacies of barrier and American options, the formulation of the utility function, and offers a complete workflow example enriched with snippets of R code.
- **Chapter 4**: This chapter presents the simulation results for each VRT and option style. It applies the decision framework in select scenarios and

extends the analysis to incorporate control variate (CV) optimization, alongside convergence and sensitivity analysis.

- **Chapter 5**: The final chapter draws conclusions from the research findings and outlines potential avenues for further investigation. It paves the way for future research, building upon the outcomes and insights derived from this study.